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Abelianization of space groups

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The abelianization of a group is its commutator quotient group. In this paper, we provide tables of the abelianizations of all the n-dimensional space groups for n = 1, 2, 3. We prove that the exponent of the torsion subgroup of the abelianization of an arbitrary n-dimensional space group Γ divides the order of the point group of Γ.

1. Introduction

An n-dimensional *crystallographic* (*space*) *group* is a discrete group Γ of isometries of Euclidean n-space E^n whose orbit space E^n/Γ is compact. As a reference for the basic theory of space groups, see Ratcliffe (2006) or Wolf (1974). For each n, there are only finitely many isomorphism types of n-dimensional space groups. The classification of the isomorphism types of space groups in low dimensions has been achieved for some time, and nowadays there are computer programs, such as *CARAT* (Opgenorth *et al.*, 1998), that will identify a low-dimensional space group given an affine representation. However, in practice, one often carries out calculations by hand, and an isomorphism invariant of a group that is easy to compute by hand is the abelianization of the group. In this paper, we provide computer-generated tables of the abelianizations of all the n-dimensional space groups for n = 1, 2, 3. We also prove some quantitative results about the abelianization of an arbitrary n-dimensional space group, which we describe below after we establish some terminology.

Let Γ be a group. The *abelianization* of Γ, denoted by Γ_{ab}, is the quotient group of Γ by its commutator subgroup $[\Gamma, \Gamma]$. The commutator subgroup $[\Gamma, \Gamma]$ is the subgroup of Γ generated by all commutators $[\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta$ of elements α, β of Γ. The commutator subgroup $[\Gamma, \Gamma]$ is a characteristic subgroup of Γ. The abelianization $\Gamma_{ab} = \Gamma/[\Gamma, \Gamma]$ is the largest Abelian (commutative) quotient of Γ. The abelianization Γ_{ab} is isomorphic to the first homology group $H_1(\Gamma, \mathbb{Z})$ of Γ with coefficients in the ring of integers $\mathbb{Z}$. The abelianizations of all the torsion-free or Coxeter n-dimensional space groups for n = 1, 2, 3 have been known for some time, but the abelianizations of the remaining two- and three-dimensional space groups have not appeared in print before to our knowledge.

The abelianization Γ_{ab} gives information about the subgroups of Γ that contain $[\Gamma, \Gamma]$. A subgroup H of Γ contains $[\Gamma, \Gamma]$ if and only if H is normal in Γ and Γ/H is Abelian. The normal subgroups of Γ with Abelian quotients correspond to the subgroups of Γ_{ab} under the quotient homomorphism from Γ to Γ_{ab}. For example, the subgroups of Γ of index two correspond to the subgroups of Γ_{ab} of index two.

Let Γ be a finitely generated group. Then $\Gamma_{ab} = G \oplus F$ where G is a finite Abelian group and F is a free Abelian group. The finite group G is a characteristic subgroup of Γ_{ab} called the *torsion subgroup* of Γ_{ab}. The rank of F is an isomorphism invariant of Γ, denoted by $\beta_1(\Gamma)$, called the *first Betti number* of Γ.

The *exponent* of a finite additive group G is the smallest positive integer e such that $eg = 0$ for all $g \in G$. The exponent of G is the least common multiple of the orders of all elements of G. In this paper, we prove that the exponent of the torsion subgroup G of the abelianization Γ_{ab} of an arbitrary n-dimensional space group Γ divides the order of the holonomy (point) group of Γ.

Let $Z(\Gamma)$ be the *center* of Γ, that is, the subgroup of all elements of Γ that commute with every element of Γ. The center $Z(\Gamma)$ is a characteristic subgroup of Γ. The following theorem, which strengthens Theorem 6 of Farkas (1981), says that the abelianization of a space group Γ determines the size of the center of Γ.

Theorem 1. [Ratcliffe & Tschantz (2008), Theorem 6.] If Γ is an n-dimensional space group, then every element of $Z(\Gamma)$ is a translation, $Z(\Gamma)$ is a free Abelian group of rank $\beta_1(\Gamma)$ and $\Gamma/Z(\Gamma)$ is an $(n - \beta_1)$-dimensional space group.

Let Γ be a space group. By Theorem 1, we have that $Z(\Gamma)$ is the subgroup of the translation group T of Γ of all the elements that are fixed by the action of Γ on T by conjugation. Therefore $Z(\Gamma)$ is the subgroup of T of all the elements that are fixed by the action of the point group Π of Γ. This implies that $\beta_1(\Gamma)$ is equal to the dimension of the fixed space of the point group Π of Γ by Theorem 1. The fixed space of Π is the intersection of the fixed spaces of a set of generators of Π, and so the first Betti number of Γ is easy to determine.

We became interested in computing Γ_{ab} when we considered the problem of identifying the space group $\Gamma/Z(\Gamma)$. By Theorem 3 of Farkas (1975), the first Betti number of $\Gamma/Z(\Gamma)$ is zero, and so the abelianization of $\Gamma/Z(\Gamma)$ is finite. This puts a restriction on the possibilities for $\Gamma/Z(\Gamma)$. For example, if $\Gamma/Z(\Gamma)$ is one-dimensional, then $\Gamma/Z(\Gamma)$ is an infinite dihedral group by consulting Table 1.

2. Computation method

We will explain our computation method by working through the computation of the abelianization of the two-dimensional space group 8-*pgg*. We start by considering generators for the group given in Table 1A of Brown *et al.* (1978). Let I denote the identity 2×2 matrix. The generators are the standard translations $\tau_1 = (1, 0) + I$ and $\tau_2 = (0, 1) + I$ together with the Euclidean isometries A and $\beta = (1/2, 1/2) + B$ where

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The standard translations generate a free Abelian group of rank two with generators τ_1 and τ_2 and defining relation $[\tau_1, \tau_2] = I$; in other words, the translation group has the group presentation $\langle \tau_1, \tau_2; [\tau_1, \tau_2] \rangle$. The point group has generators A, B subject to the relations $A^2 = I$, $B^2 = I$, $[A, B] = I$ given in Table 1A of Brown *et al.* (1978). Thus the point group of 8-*pgg* has the group presentation

$$\langle A, B; A^2, B^2, [A, B] \rangle.$$

We now build a group presentation for the space group 8-*pgg* by lifting the generators and relations of the point group and adding relations that show how the lifted generators act on the standard translations by conjugation. The generators A and B of the point group lift to generators A and β, respectively. The relations $A^2 = I$, $B^2 = I$, $[A, B] = I$ of the point group lift to the relations $A^2 = I$, $\beta^2 = \tau_1$, $[A, \beta] = \tau_1 \tau_2^{-1}$, respectively. The matrices A and B determine how the generators A and β act on τ_1 and τ_2 by conjugation. For example, the columns of B imply that $\beta\tau_1\beta^{-1} = \tau_1$ and $\beta\tau_2\beta^{-1} = \tau_2^{-1}$. We obtain the group presentation for the two-dimensional space group 8-*pgg*,

$$\langle \tau_1, \tau_2, A, \beta; [\tau_1, \tau_2], A^2, \beta^2\tau_1^{-1}, [A, \beta]\tau_1^{-1}\tau_2, A\tau_1A^{-1}\tau_1, \\ A\tau_2A^{-1}\tau_2, \beta\tau_1\beta^{-1}\tau_1^{-1}, \beta\tau_2\beta^{-1}\tau_2 \rangle.$$

We next abelianize the above presentation by making all generators commute. This gives the Abelian group presentation for the abelianization of 8-*pgg*,

$$\langle \tau_1, \tau_2, A, \beta; A^2, \beta^2\tau_1^{-1}, \tau_1^{-1}\tau_2, \tau_1^2, \tau_2^2 \rangle.$$

We simplify the above presentation by eliminating the generator τ_1 *via* the relation $\tau_1 = \tau_2$ derived from the relator $\tau_1^{-1}\tau_2$, and obtain the Abelian group presentation

$$\langle \tau_2, A, \beta; A^2, \beta^2\tau_2^{-1}, \tau_2^2 \rangle.$$

We next eliminate the generator τ_2 *via* the relation $\beta^2 = \tau_2$ and obtain the Abelian group presentation $\langle A, \beta; A^2, \beta^4 \rangle$. Hence the abelianization of the two-dimensional space group 8-*pgg* is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4$. Here $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the additive cyclic group of order n.

In our general computer computation, we skip all the simplification of the presentations and apply a standard linear algebra method, discussed in §3.3 of Magnus *et al.* (1966), to compute the abelianization of a space group from an initial group presentation of the space group.

We now explain this method by using it to compute the abelianization of 8-*pgg*. The method begins by forming the presentation $m \times n$ matrix M, for the Abelian group presentation, whose ijth entry is the exponent of the ith generator in the jth relator. For our example, we obtain the following 4×5 matrix shown with row and column headings:

	A^2	$\beta^2\tau_1^{-1}$	$\tau_1^{-1}\tau_2$	τ_1^2	τ_2^2
τ_1	0	−1	−1	2	0
τ_2	0	0	1	0	2
A	2	0	0	0	0
β	0	2	0	0	0

The idea of the method is to perform a sequence of elementary row and column operations on M to convert M into a matrix of the form

$$\left(\begin{array}{ccc|c} d_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & d_k & \\ \hline & 0 & & 0 \end{array}\right)$$

with $d_i \geq 1$ for each i. This implies that M presents the Abelian group

$$\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k} \oplus \mathbb{Z}^{m-k}.$$

The elementary operations allowed on M are

(1) interchange two rows (columns),

(2) change the sign of a row (column),

(3) add an integral multiple of a row (column) to another row (column).

The first step is to move the smallest nonzero entry of M in absolute value to the upper left-hand corner by type (1) elementary operations. For our example, we interchange columns 1 and 2 to obtain the matrix

$$\begin{pmatrix} -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The next step is to make the new upper left-hand corner entry positive by a type (2) elementary operation. For our example, we multiply the first row by −1 to obtain the matrix

$$\begin{pmatrix} 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The next step is to use the new upper left-hand corner entry to reduce the entries of the first row and first column by type (3) elementary operations. For our example, we subtract the first column from the third column and we add twice the first column to the fourth column to obtain the matrix

Table 1
The abelianizations of the one-dimensional space groups.

IT	BBNWZ	Group	Conway	*CARAT*	Abelianization	Holonomy
1	1/1/1/1	C_∞	()	min.1-1.1-0	$\mathbb{Z}$	C_1
2	1/2/1/1	D_∞	[]	max.1-1.1-0	$\mathbb{Z}_2^2$	C_2

Table 2
The abelianizations of the two-dimensional space groups.

IT	BBNWZ	HM	Conway	*CARAT*	Abelianization	Holonomy
1	1/1/1/1	*p*1	∘	min.2-1.1-0	$\mathbb{Z}^2$	C_1
2	1/2/1/1	*p*2	2222	group.1-1.1-0	$\mathbb{Z}_2^3$	C_2
3	2/1/1/1	*pm*	∗∗	min.3-1.1-0	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_2
4	2/1/1/2	*pg*	××	min.3-1.1-1	$\mathbb{Z} \oplus \mathbb{Z}_2$	C_2
5	2/1/2/1	*cm*	∗×	min.3-1.2-0	$\mathbb{Z} \oplus \mathbb{Z}_2$	C_2
6	2/2/1/1	*pmm*	∗2222	group.2-1.1-0	$\mathbb{Z}_2^4$	C_2^2
7	2/2/1/2	*pmg*	22∗	group.2-1.1-1	$\mathbb{Z}_2^3$	C_2^2
8	2/2/1/3	*pgg*	22×	group.2-1.1-3	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	C_2^2
9	2/2/2/1	*cmm*	2∗22	group.2-1.2-0	$\mathbb{Z}_2^3$	C_2^2
10	3/1/1/1	*p*4	442	min.4-1.1-0	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	C_4
11	3/2/1/1	*p*4*m*	∗442	max.2-1.1-0	$\mathbb{Z}_2^3$	D_4
12	3/2/1/2	*p*4*g*	4∗2	max.2-1.1-1	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	D_4
13	4/1/1/1	*p*3	333	min.5-1.1-0	$\mathbb{Z}_3^2$	C_3
14	4/2/1/1	*p*3*m*1	∗333	group.4-2.1-0	$\mathbb{Z}_2$	D_3
15	4/2/2/1	*p*31*m*	3∗3	group.4-1.1-0	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	D_3
16	4/3/1/1	*p*6	632	group.3-1.1-0	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	C_6
17	4/4/1/1	*p*6*m*	∗632	max.3-1.1-0	$\mathbb{Z}_2^2$	D_6

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & -2 & 4 & 0 \end{pmatrix}.$$

Then we subtract twice the first row from the fourth row to obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 & 0 \end{pmatrix}.$$

If in the above process we create a nonzero entry that is smaller in absolute value than the upper left-hand corner entry, then we go back to the first step and repeat the process. If the upper left-hand corner entry is still a smallest nonzero entry in absolute value, then all the other entries in the first row and first column are zero. In this case, ignore the first row and first column and repeat the above process with the remaining $(m-1) \times (n-1)$ submatrix. For our example, we interchange columns 2 and 3 to obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 4 & 0 \end{pmatrix}.$$

We subtract twice column 2 from column 4 to obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 4 & 4 \end{pmatrix}.$$

Then we add twice row 2 to row 4 to obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 \end{pmatrix}.$$

The final step for our example is to subtract column 4 from column 5 to obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix},$$

which is of the desired form. From this final matrix, we see that the abelianization of 8-*pgg* is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4$. The 1 entries of the matrix do not contribute anything, since $\mathbb{Z}_1 = \mathbb{Z}/\mathbb{Z} = \{0\}$.

3. Tables

In order to make our tables as useful as possible, we have included all the most common names for the the space groups. The column headed IT gives the *International Tables* (Hahn,

Table 3
The abelianizations of the three-dimensional space groups.

IT	BBNWZ	HM	Conway	*CARAT*	Abelianization	Holonomy
1	1/1/1/1	$P1$	$(\circ)$	min.6-1.1-0	$\mathbb{Z}^3$	C_1
2	1/2/1/1	$P\bar{1}$	(2222)	group.5-1.1-0	$\mathbb{Z}_2^4$	C_2
3	2/1/1/1	$P2$	$(2_0 2_0 2_0 2_0)$	min.7-1.1-0	$\mathbb{Z} \oplus \mathbb{Z}_2^3$	C_2
4	2/1/1/2	$P2_1$	$(2_1 2_1 2_1 2_1)$	min.7-1.1-1	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_2
5	2/1/2/1	$C2$	$(2_0 2_0 2_1 2_1)$	min.7-1.2-0	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_2
6	2/2/1/1	Pm	$[\circ_0]$	min.8-1.1-0	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$	C_2
7	2/2/1/2	Pc	$(\bar{\circ}_0)$	min.8-1.1-1	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	C_2
8	2/2/2/1	Cm	$[\circ_1]$	min.8-1.2-0	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	C_2
9	2/2/2/2	Cc	$(\bar{\circ}_1)$	min.8-1.2-1	$\mathbb{Z}^2$	C_2
10	2/3/1/1	$P2/m$	$[2_0 2_0 2_0 2_0]$	group.6-1.1-0	$\mathbb{Z}_2^5$	C_2^2
11	2/3/1/3	$P2_1/m$	$[2_1 2_1 2_1 2_1]$	group.6-1.1-1	$\mathbb{Z}_2^4$	C_2^2
12	2/3/2/1	$C2/m$	$[2_0 2_0 2_1 2_1]$	group.6-1.2-0	$\mathbb{Z}_2^4$	C_2^2
13	2/3/1/2	$P2/c$	$(2_0 2_0 22)$	group.6-1.1-2	$\mathbb{Z}_2^4$	C_2^2
14	2/3/1/4	$P2_1/c$	$(2_1 2_1 22)$	group.6-1.1-3	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	C_2^2
15	2/3/2/2	$C2/c$	$(2_0 2_1 22)$	group.6-1.2-1	$\mathbb{Z}_2^3$	C_2^2
16	3/1/1/1	$P222$	$(\ast 2_0 2_0 2_0 2_0)$	min.10-1.1-0	$\mathbb{Z}_2^5$	C_2^2
17	3/1/1/2	$P222_1$	$(\ast 2_1 2_1 2_1 2_1)$	min.10-1.1-1	$\mathbb{Z}_2^4$	C_2^2
18	3/1/1/3	$P2_1 2_1 2$	$(2_0 2_0 \bar{\times})$	min.10-1.1-3	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	C_2^2
19	3/1/1/4	$P2_1 2_1 2_1$	$(2_1 2_1 \bar{\times})$	min.10-1.1-7	$\mathbb{Z}_4^2$	C_2^2
20	3/1/2/2	$C222_1$	$(2_1 \ast 2_1 2_1)$	min.10-1.2-1	$\mathbb{Z}_2^3$	C_2^2
21	3/1/2/1	$C222$	$(2_0 \ast 2_0 2_0)$	min.10-1.2-0	$\mathbb{Z}_2^4$	C_2^2
22	3/1/3/1	$F222$	$(\ast 2_0 2_1 2_0 2_1)$	min.10-1.3-0	$\mathbb{Z}_2^4$	C_2^2
23	3/1/4/1	$I222$	$(2_1 \ast 2_0 2_0)$	min.10-1.4-0	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	C_2^2
24	3/1/4/2	$I2_1 2_1 2_1$	$(2_0 \ast 2_1 2_1)$	min.10-1.4-1	$\mathbb{Z}_2^3$	C_2^2
25	3/2/1/1	$Pmm2$	$(\ast{\cdot}2{\cdot}2{\cdot}2{\cdot}2)$	min.9-1.1-0	$\mathbb{Z} \oplus \mathbb{Z}_2^4$	C_2^2
26	3/2/1/7	$Pmc2_1$	$(\ast{\cdot}2{:}2{\cdot}2{:}2)$	min.9-1.1-2	$\mathbb{Z} \oplus \mathbb{Z}_2^3$	C_2^2
27	3/2/1/3	$Pcc2$	$(\ast{:}2{:}2{:}2{:}2)$	min.9-1.1-10	$\mathbb{Z} \oplus \mathbb{Z}_2^3$	C_2^2
28	3/2/1/2	$Pma2$	$(2_0 2_0 \ast{\cdot})$	min.9-1.1-1	$\mathbb{Z} \oplus \mathbb{Z}_2^3$	C_2^2
29	3/2/1/9	$Pca2_1$	$(2_1 2_1 \ast{:})$	min.9-1.1-6	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_2^2
30	3/2/1/4	$Pnc2$	$(2_0 2_0 \ast{:})$	min.9-1.1-11	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_2^2
31	3/2/1/8	$Pmn2_1$	$(2_1 2_1 \ast{\cdot})$	min.9-1.1-3	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_2^2
32	3/2/1/5	$Pba2$	$(2_0 2_0 \times_0)$	min.9-1.1-5	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	C_2^2
33	3/2/1/10	$Pna2_1$	$(2_1 2_1 \times)$	min.9-1.1-7	$\mathbb{Z} \oplus \mathbb{Z}_4$	C_2^2
34	3/2/1/6	$Pnn2$	$(2_0 2_0 \times_1)$	min.9-1.1-15	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_2^2
35	3/2/2/1	$Cmm2$	$(2_0 \ast{\cdot}2{\cdot}2)$	min.9-1.2-0	$\mathbb{Z} \oplus \mathbb{Z}_2^3$	C_2^2
36	3/2/2/3	$Cmc2_1$	$(2_1 \ast{\cdot}2{:}2)$	min.9-1.2-1	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_2^2
37	3/2/2/2	$Ccc2$	$(2_0 \ast{:}2{:}2)$	min.9-1.2-3	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_2^2
38	3/2/3/1	$Amm2$	$(\ast{\cdot}2{\cdot}2{\cdot}2{:}2)$	min.9-1.3-0	$\mathbb{Z} \oplus \mathbb{Z}_2^3$	C_2^2
39	3/2/3/2	$Abm2$	$(\ast{\cdot}2{:}2{:}2{:}2)$	min.9-1.3-2	$\mathbb{Z} \oplus \mathbb{Z}_2^3$	C_2^2
40	3/2/3/3	$Ama2$	$(2_0 2_1 \ast{\cdot})$	min.9-1.3-1	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_2^2
41	3/2/3/4	$Aba2$	$(2_0 2_1 \ast{:})$	min.9-1.3-3	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_2^2
42	3/2/4/1	$Fmm2$	$(\ast{\cdot}2{\cdot}2{:}2{:}2)$	min.9-1.4-0	$\mathbb{Z} \oplus \mathbb{Z}_2^3$	C_2^2
43	3/2/4/2	$Fdd2$	$(2_0 2_1 \times)$	min.9-1.4-1	$\mathbb{Z} \oplus \mathbb{Z}_2$	C_2^2
44	3/2/5/1	$Imm2$	$(2_1 \ast{\cdot}2{\cdot}2)$	min.9-1.5-0	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_2^2
45	3/2/5/3	$Iba2$	$(2_1 \ast{:}2{:}2)$	min.9-1.5-3	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_2^2
46	3/2/5/2	$Ima2$	$(2_0 \ast{\cdot}2{:}2)$	min.9-1.5-1	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_2^2
47	3/3/1/1	$Pmmm$	$[\ast{\cdot}2{\cdot}2{\cdot}2{\cdot}2]$	group.7-1.1-0	$\mathbb{Z}_2^6$	C_2^3
48	3/3/1/4	$Pnnn$	$(2\bar{\ast}_1 2_0 2_0)$	group.7-1.1-63	$\mathbb{Z}_2^4$	C_2^3
49	3/3/1/2	$Pccm$	$[\ast{:}2{:}2{:}2{:}2]$	group.7-1.1-10	$\mathbb{Z}_2^5$	C_2^3
50	3/3/1/3	$Pban$	$(2\bar{\ast}_0 2_0 2_0)$	group.7-1.1-30	$\mathbb{Z}_2^4$	C_2^3
51	3/3/1/5	$Pmma$	$[2_0 2_0 \ast{\cdot}]$	group.7-1.1-1	$\mathbb{Z}_2^5$	C_2^3
52	3/3/1/8	$Pnna$	$(2_0 2\bar{\ast}_1)$	group.7-1.1-31	$\mathbb{Z}_2^3$	C_2^3
53	3/3/1/7	$Pmna$	$[2_0 2_0 \ast{:}]$	group.7-1.1-11	$\mathbb{Z}_2^4$	C_2^3
54	3/3/1/6	$Pcca$	$(2_0 2\bar{\ast}_0)$	group.7-1.1-21	$\mathbb{Z}_2^4$	C_2^3
55	3/3/1/13	$Pbam$	$[2_0 2_0 \times_0]$	group.7-1.1-5	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_4$	C_2^3

Table 3 (continued)

IT	BBNWZ	HM	Conway	*CARAT*	Abelianization	Holonomy
56	3/3/1/12	*Pccn*	$(2\bar{*}:2:2)$	group.7-1.1-23	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	C_2^3
57	3/3/1/9	*Pbcm*	$(2_02\bar{*}\cdot)$	group.7-1.1-6	$\mathbb{Z}_2^4$	C_2^3
58	3/3/1/14	*Pnnm*	$[2_02_0\times_1]$	group.7-1.1-15	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	C_2^3
59	3/3/1/11	*Pmmn*	$(2\bar{*}\cdot2\cdot2)$	group.7-1.1-3	$\mathbb{Z}_2^4$	C_2^3
60	3/3/1/10	*Pbcn*	$(2_02\bar{*}:)$	group.7-1.1-27	$\mathbb{Z}_2^3$	C_2^3
61	3/3/1/16	*Pbca*	$(2_12\bar{*}:)$	group.7-1.1-25	$\mathbb{Z}_2^3$	C_2^3
62	3/3/1/15	*Pnma*	$(2_12\bar{*}\cdot)$	group.7-1.1-7	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	C_2^3
63	3/3/2/3	*Cmcm*	$[2_02_1*\cdot]$	group.7-1.2-1	$\mathbb{Z}_2^4$	C_2^3
64	3/3/2/6	*Cmca*	$[2_02_1*:]$	group.7-1.2-5	$\mathbb{Z}_2^4$	C_2^3
65	3/3/2/1	*Cmmm*	$[2_0*\cdot2\cdot2]$	group.7-1.2-0	$\mathbb{Z}_2^5$	C_2^3
66	3/3/2/2	*Cccm*	$[2_0*:2:2]$	group.7-1.2-3	$\mathbb{Z}_2^4$	C_2^3
67	3/3/2/4	*Cmma*	$(*2_02\cdot2\cdot2)$	group.7-1.2-4	$\mathbb{Z}_2^5$	C_2^3
68	3/3/2/5	*Ccca*	$(*2_02:2:2)$	group.7-1.2-7	$\mathbb{Z}_2^4$	C_2^3
69	3/3/3/1	*Fmmm*	$[*\cdot2\cdot2:2:2]$	group.7-1.3-0	$\mathbb{Z}_2^5$	C_2^3
70	3/3/3/2	*Fddd*	$(2\bar{*}2_02_1)$	group.7-1.3-1	$\mathbb{Z}_2^3$	C_2^3
71	3/3/4/1	*Immm*	$[2_1*\cdot2\cdot2]$	group.7-1.4-0	$\mathbb{Z}_2^4$	C_2^3
72	3/3/4/2	*Ibam*	$[2_1*:2:2]$	group.7-1.4-3	$\mathbb{Z}_2^4$	C_2^3
73	3/3/4/4	*Ibca*	$(*2_12:2:2)$	group.7-1.4-7	$\mathbb{Z}_2^4$	C_2^3
74	3/3/4/3	*Imma*	$(*2_12\cdot2\cdot2)$	group.7-1.4-1	$\mathbb{Z}_2^4$	C_2^3
75	4/1/1/1	*P*4	$(4_04_02_0)$	min.11-1.1-0	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	C_4
76	4/1/1/2	$P4_1$	$(4_14_12_1)$	min.11-1.1-1	$\mathbb{Z} \oplus \mathbb{Z}_2$	C_4
77	4/1/1/3	$P4_2$	$(4_24_22_0)$	min.11-1.1-2	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	C_4
79	4/1/2/1	*I*4	$(4_24_02_1)$	min.11-1.2-0	$\mathbb{Z} \oplus \mathbb{Z}_4$	C_4
80	4/1/2/2	$I4_1$	$(4_34_12_0)$	min.11-1.2-1	$\mathbb{Z} \oplus \mathbb{Z}_2$	C_4
81	4/2/1/1	$P\bar{4}$	(442_0)	min.12-1.1-0	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	C_4
82	4/2/2/1	$I\bar{4}$	(442_1)	min.12-1.2-0	$\mathbb{Z}_4^2$	C_4
83	4/3/1/1	$P4/m$	$[4_04_02_0]$	group.12-1.1-0	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_4$	$C_4 \times C_2$
84	4/3/1/2	$P4_2/m$	$[4_24_22_0]$	group.12-1.1-2	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	$C_4 \times C_2$
85	4/3/1/3	$P4/n$	(44_02)	group.12-1.1-1	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	$C_4 \times C_2$
86	4/3/1/4	$P4_2/n$	(44_22)	group.12-1.1-3	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	$C_4 \times C_2$
87	4/3/2/1	$I4/m$	$[4_24_02_1]$	group.12-1.2-0	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	$C_4 \times C_2$
88	4/3/2/2	$I4_1/a$	(44_12)	group.12-1.2-1	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	$C_4 \times C_2$
89	4/4/1/1	*P*422	$(*4_04_02_0)$	group.11-1.1-0	$\mathbb{Z}_2^4$	D_4
90	4/4/1/4	$P42_12$	(4_0*2_0)	group.11-1.1-1	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	D_4
91	4/4/1/2	$P4_122$	$(*4_14_12_1)$	group.11-1.1-2	$\mathbb{Z}_2^3$	D_4
92	4/4/1/5	$P4_12_12$	(4_1*2_1)	group.11-1.1-3	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	D_4
93	4/4/1/3	$P4_222$	$(*4_24_22_0)$	group.11-1.1-4	$\mathbb{Z}_2^4$	D_4
94	4/4/1/6	$P4_22_12$	(4_2*2_0)	group.11-1.1-5	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	D_4
97	4/4/2/1	*I*422	$(*4_24_02_1)$	group.11-1.2-0	$\mathbb{Z}_2^3$	D_4
98	4/4/2/2	$I4_122$	$(*4_34_12_0)$	group.11-1.2-1	$\mathbb{Z}_2^3$	D_4
99	4/5/1/1	*P4mm*	$(*\cdot4\cdot4\cdot2)$	group.10-1.1-0	$\mathbb{Z} \oplus \mathbb{Z}_2^3$	D_4
100	4/5/1/5	*P4bm*	$(4_0*\cdot2)$	group.10-1.1-2	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	D_4
101	4/5/1/4	$P4_2cm$	$(*:4\cdot4:2)$	group.10-1.1-4	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	D_4
102	4/5/1/8	$P4_2nm$	$(4_2*\cdot2)$	group.10-1.1-6	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	D_4
103	4/5/1/3	*P4cc*	$(*:4:4:2)$	group.10-1.1-5	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	D_4
104	4/5/1/7	*P4nc*	$(4_0*:2)$	group.10-1.1-7	$\mathbb{Z} \oplus \mathbb{Z}_4$	D_4
105	4/5/1/2	$P4_2mc$	$(*\cdot4:4\cdot2)$	group.10-1.1-1	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	D_4
106	4/5/1/6	$P4_2bc$	$(4_2*:2)$	group.10-1.1-3	$\mathbb{Z} \oplus \mathbb{Z}_4$	D_4
107	4/5/2/1	*I4mm*	$(*\cdot4\cdot4:2)$	group.10-1.2-0	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	D_4
108	4/5/2/2	*I4cm*	$(*\cdot4:4:2)$	group.10-1.2-1	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	D_4
109	4/5/2/3	$I4_1md$	$(4_1*\cdot2)$	group.10-1.2-2	$\mathbb{Z} \oplus \mathbb{Z}_2$	D_4
110	4/5/2/4	$I4_1cd$	$(4_1*:2)$	group.10-1.2-3	$\mathbb{Z} \oplus \mathbb{Z}_2$	D_4
111	4/6/1/1	$P\bar{4}2m$	$(*4\cdot42_0)$	group.9-2.1-0	$\mathbb{Z}_2^4$	D_4
112	4/6/1/2	$P\bar{4}2c$	$(*4:42_0)$	group.9-2.1-2	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	D_4
113	4/6/1/3	$P\bar{4}2_1m$	$(4\bar{*}\cdot2)$	group.9-2.1-1	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	D_4
114	4/6/1/4	$P\bar{4}2_1c$	$(4\bar{*}:2)$	group.9-2.1-3	$\mathbb{Z}_4^2$	D_4

Table 3 (continued)

IT	BBNWZ	HM	Conway	*CARAT*	Abelianization	Holonomy
115	4/6/2/1	$P\bar{4}m2$	$(\ast{\cdot}44{\cdot}2)$	group.9-1.1-0	$\mathbb{Z}_2^4$	D_4
116	4/6/2/2	$P\bar{4}c2$	$(\ast{:}44{:}2)$	group.9-1.1-2	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	D_4
117	4/6/2/3	$P\bar{4}b2$	$(4\bar{\ast}_0 2_0)$	group.9-1.1-1	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	D_4
118	4/6/2/4	$P\bar{4}n2$	$(4\bar{\ast}_1 2_0)$	group.9-1.1-3	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	D_4
119	4/6/3/1	$I\bar{4}m2$	$(\ast 4{\cdot}42_1)$	group.9-2.2-0	$\mathbb{Z}_2^3$	D_4
120	4/6/3/2	$I\bar{4}c2$	$(\ast 4{:}42_1)$	group.9-2.2-1	$\mathbb{Z}_2^3$	D_4
121	4/6/4/1	$I\bar{4}2m$	$(\ast{\cdot}44{:}2)$	group.9-1.2-0	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	D_4
122	4/6/4/2	$I\bar{4}2d$	$(4\bar{\ast}2_1)$	group.9-1.2-1	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	D_4
123	4/7/1/1	$P4/mmm$	$[\ast{\cdot}4{\cdot}4{\cdot}2]$	group.8-1.1-0	$\mathbb{Z}_2^5$	$D_4 \times C_2$
124	4/7/1/2	$P4/mcc$	$[\ast{:}4{:}4{:}2]$	group.8-1.1-9	$\mathbb{Z}_2^4$	$D_4 \times C_2$
125	4/7/1/5	$P4/nbm$	$(\ast 4_0 4{\cdot}2)$	group.8-1.1-2	$\mathbb{Z}_2^4$	$D_4 \times C_2$
126	4/7/1/6	$P4/nnc$	$(\ast 4_0 4{:}2)$	group.8-1.1-11	$\mathbb{Z}_2^3$	$D_4 \times C_2$
127	4/7/1/13	$P4/mbm$	$[4_0 \ast{\cdot}2]$	group.8-1.1-6	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_4$	$D_4 \times C_2$
128	4/7/1/14	$P4/mnc$	$[4_0 \ast{:}2]$	group.8-1.1-15	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	$D_4 \times C_2$
129	4/7/1/9	$P4/nmm$	$(\ast 4{\cdot}4{\cdot}2)$	group.8-1.1-4	$\mathbb{Z}_2^4$	$D_4 \times C_2$
130	4/7/1/10	$P4/ncc$	$(\ast 4{:}4{:}2)$	group.8-1.1-13	$\mathbb{Z}_2^3$	$D_4 \times C_2$
131	4/7/1/3	$P4_2/mmc$	$[\ast{\cdot}4{:}4{\cdot}2]$	group.8-1.1-8	$\mathbb{Z}_2^4$	$D_4 \times C_2$
132	4/7/1/4	$P4_2/mcm$	$[\ast{:}4{\cdot}4{:}2]$	group.8-1.1-1	$\mathbb{Z}_2^4$	$D_4 \times C_2$
133	4/7/1/7	$P4_2/nbc$	$(\ast 4_2 4{:}2)$	group.8-1.1-10	$\mathbb{Z}_2^3$	$D_4 \times C_2$
134	4/7/1/8	$P4_2/nnm$	$(\ast 4_2 4{\cdot}2)$	group.8-1.1-3	$\mathbb{Z}_2^4$	$D_4 \times C_2$
135	4/7/1/15	$P4_2/mbc$	$[4_2 \ast{:}2]$	group.8-1.1-14	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	$D_4 \times C_2$
136	4/7/1/16	$P4_2/mnm$	$[4_2 \ast{\cdot}2]$	group.8-1.1-7	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	$D_4 \times C_2$
137	4/7/1/11	$P4_2/nmc$	$(\ast 4{\cdot}4{:}2)$	group.8-1.1-12	$\mathbb{Z}_2^3$	$D_4 \times C_2$
138	4/7/1/12	$P4_2/ncm$	$(\ast 4{:}4{\cdot}2)$	group.8-1.1-5	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	$D_4 \times C_2$
139	4/7/2/1	$I4/mmm$	$[\ast{\cdot}4{\cdot}4{:}2]$	group.8-1.2-0	$\mathbb{Z}_2^4$	$D_4 \times C_2$
140	4/7/2/2	$I4/mcm$	$[\ast{\cdot}4{:}4{:}2]$	group.8-1.2-2	$\mathbb{Z}_2^4$	$D_4 \times C_2$
141	4/7/2/3	$I4_1/amd$	$(\ast 4_1 4{\cdot}2)$	group.8-1.2-1	$\mathbb{Z}_2^3$	$D_4 \times C_2$
142	4/7/2/4	$I4_1/acd$	$(\ast 4_1 4{:}2)$	group.8-1.2-3	$\mathbb{Z}_2^3$	$D_4 \times C_2$
143	5/1/2/1	$P3$	$(3_0 3_0 3_0)$	min.13-1.1-0	$\mathbb{Z} \oplus \mathbb{Z}_3^2$	C_3
144	5/1/2/2	$P3_1$	$(3_1 3_1 3_1)$	min.13-1.1-1	$\mathbb{Z} \oplus \mathbb{Z}_3$	C_3
146	5/1/1/1	$R3$	$(3_0 3_1 3_2)$	min.13-1.2-0	$\mathbb{Z} \oplus \mathbb{Z}_3$	C_3
147	5/2/2/1	$P\bar{3}$	$(63_0 2)$	group.22-1.1-0	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	C_6
148	5/2/1/1	$R\bar{3}$	$(63_1 2)$	group.22-1.2-0	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	C_6
149	5/3/2/1	$P312$	$(\ast 3_0 3_0 3_0)$	group.20-1.1-0	$\mathbb{Z}_2^2$	D_3
150	5/3/3/1	$P321$	$(3_0 \ast 3_0)$	group.20-2.1-0	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	D_3
151	5/3/2/2	$P3_1 12$	$(\ast 3_1 3_1 3_1)$	group.20-1.1-1	$\mathbb{Z}_2^2$	D_3
152	5/3/3/2	$P3_1 21$	$(3_1 \ast 3_1)$	group.20-2.1-1	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	D_3
155	5/3/1/1	$R32$	$(\ast 3_0 3_1 3_2)$	group.20-1.2-0	$\mathbb{Z}_2^2$	D_3
156	5/4/2/1	$P3m1$	$(\ast{\cdot}3{\cdot}3{\cdot}3)$	group.21-2.1-0	$\mathbb{Z} \oplus \mathbb{Z}_2$	D_3
157	5/4/3/1	$P31m$	$(3_0 \ast{\cdot}3)$	group.21-1.1-0	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	D_3
158	5/4/2/2	$P3c1$	$(\ast{:}3{:}3{:}3)$	group.21-2.1-1	$\mathbb{Z}$	D_3
159	5/4/3/2	$P31c$	$(3_0 \ast{:}3)$	group.21-1.1-1	$\mathbb{Z} \oplus \mathbb{Z}_3$	D_3
160	5/4/1/1	$R3m$	$(3_1 \ast{\cdot}3)$	group.21-1.2-0	$\mathbb{Z} \oplus \mathbb{Z}_2$	D_3
161	5/4/1/2	$R3c$	$(3_1 \ast{:}3)$	group.21-1.2-1	$\mathbb{Z}$	D_3
162	5/5/2/1	$P\bar{3}1m$	$(\ast{\cdot}63_0 2)$	group.14-1.1-0	$\mathbb{Z}_2^3$	D_6
163	5/5/2/2	$P\bar{3}1c$	$(\ast{:}63_0 2)$	group.14-1.1-1	$\mathbb{Z}_2^2$	D_6
164	5/5/3/1	$P\bar{3}m1$	$(\ast 6{\cdot}3{\cdot}2)$	group.14-2.1-0	$\mathbb{Z}_2^3$	D_6
165	5/5/3/2	$P\bar{3}c1$	$(\ast 6{:}3{:}2)$	group.14-2.1-1	$\mathbb{Z}_2^2$	D_6
166	5/5/1/1	$R\bar{3}m$	$(\ast{\cdot}63_1 2)$	group.14-1.2-0	$\mathbb{Z}_2^3$	D_6
167	5/5/1/2	$R\bar{3}c$	$(\ast{:}63_1 2)$	group.14-1.2-1	$\mathbb{Z}_2^2$	D_6
168	6/1/1/1	$P6$	$(6_0 3_0 2_0)$	group.18-1.1-0	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	C_6
169	6/1/1/4	$P6_1$	$(6_1 3_1 2_1)$	group.18-1.1-1	$\mathbb{Z}$	C_6
171	6/1/1/2	$P6_2$	$(6_2 3_2 2_0)$	group.18-1.1-2	$\mathbb{Z} \oplus \mathbb{Z}_2$	C_6
173	6/1/1/3	$P6_3$	$(6_3 3_0 2_1)$	group.18-1.1-3	$\mathbb{Z} \oplus \mathbb{Z}_3$	C_6
174	6/2/1/1	$P\bar{6}$	$[3_0 3_0 3_0]$	group.19-1.1-0	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3^2$	C_6
175	6/3/1/1	$P6/m$	$[6_0 3_0 2_0]$	group.13-1.1-0	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3$	$C_6 \times C_2$

Table 3 (continued)

IT	BBNWZ	HM	Conway	*CARAT*	Abelianization	Holonomy
176	6/3/1/2	$P6_3/m$	$[6_3 3_0 2_1]$	group.13-1.1-1	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	$C_6 \times C_2$
177	6/4/1/1	$P622$	$(*6_0 3_0 2_0)$	group.17-1.1-0	$\mathbb{Z}_2^3$	D_6
178	6/4/1/4	$P6_122$	$(*6_1 3_1 2_1)$	group.17-1.1-1	$\mathbb{Z}_2^2$	D_6
180	6/4/1/2	$P6_222$	$(*6_2 3_2 2_0)$	group.17-1.1-2	$\mathbb{Z}_2^3$	D_6
182	6/4/1/3	$P6_322$	$(*6_3 3_0 2_1)$	group.17-1.1-3	$\mathbb{Z}_2^2$	D_6
183	6/5/1/1	$P6mm$	$(*{\cdot}6{\cdot}3{\cdot}2)$	group.15-1.1-0	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	D_6
184	6/5/1/2	$P6cc$	$(*{:}6{:}3{:}2)$	group.15-1.1-3	$\mathbb{Z} \oplus \mathbb{Z}_2$	D_6
185	6/5/1/4	$P6_3cm$	$(*{\cdot}6{:}3{:}2)$	group.15-1.1-2	$\mathbb{Z} \oplus \mathbb{Z}_2$	D_6
186	6/5/1/3	$P6_3mc$	$(*{:}6{\cdot}3{\cdot}2)$	group.15-1.1-1	$\mathbb{Z} \oplus \mathbb{Z}_2$	D_6
187	6/6/1/1	$P\bar{6}m2$	$[*{\cdot}3{\cdot}3{\cdot}3]$	group.16-1.1-0	$\mathbb{Z}_2^3$	D_6
188	6/6/1/2	$P\bar{6}c2$	$[*{:}3{:}3{:}3]$	group.16-1.1-1	$\mathbb{Z}_2^2$	D_6
189	6/6/2/1	$P\bar{6}2m$	$[3_0*{\cdot}3]$	group.16-2.1-0	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3$	D_6
190	6/6/2/2	$P\bar{6}2c$	$[3_0*{:}3]$	group.16-2.1-1	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	D_6
191	6/7/1/1	$P6/mmm$	$[*{\cdot}6{\cdot}3{\cdot}2]$	max.4-1.1-0	$\mathbb{Z}_2^4$	$D_6 \times C_2$
192	6/7/1/2	$P6/mcc$	$[*{:}6{:}3{:}2]$	max.4-1.1-3	$\mathbb{Z}_2^3$	$D_6 \times C_2$
193	6/7/1/4	$P6_3/mcm$	$[*{\cdot}6{:}3{:}2]$	max.4-1.1-2	$\mathbb{Z}_2^3$	$D_6 \times C_2$
194	6/7/1/3	$P6_3/mmc$	$[*{:}6{\cdot}3{\cdot}2]$	max.4-1.1-1	$\mathbb{Z}_2^3$	$D_6 \times C_2$
195	7/1/1/1	$P23$	2°	min.14-2.1-0	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	A_4
196	7/1/2/1	$F23$	1°	min.14-1.1-0	$\mathbb{Z}_3$	A_4
197	7/1/3/1	$I23$	$4^{\circ\circ}$	min.14-3.1-0	$\mathbb{Z}_3 \oplus \mathbb{Z}_4$	A_4
198	7/1/1/2	$P2_13$	$1^{\circ}/4$	min.14-2.1-1	$\mathbb{Z}_3$	A_4
199	7/1/3/2	$I2_13$	$2^{\circ}/4$	min.14-3.1-1	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	A_4
200	7/2/1/1	$Pm\bar{3}$	4^{-}	group.23-2.1-0	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	$A_4 \times C_2$
201	7/2/1/2	$Pn\bar{3}$	$4^{\circ+}$	group.23-2.1-1	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	$A_4 \times C_2$
202	7/2/2/1	$Fm\bar{3}$	2^{-}	group.23-1.1-0	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	$A_4 \times C_2$
203	7/2/2/2	$Fd\bar{3}$	$2^{\circ+}$	group.23-1.1-1	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	$A_4 \times C_2$
204	7/2/3/1	$Im\bar{3}$	$8^{-\circ}$	group.23-3.1-0	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	$A_4 \times C_2$
205	7/2/1/3	$Pa\bar{3}$	$2^{-}/4$	group.23-2.1-2	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	$A_4 \times C_2$
206	7/2/3/2	$Ia\bar{3}$	$4^{-}/4$	group.23-3.1-1	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	$A_4 \times C_2$
207	7/3/1/1	$P432$	$4^{\circ-}$	group.25-2.1-0	$\mathbb{Z}_2^2$	S_4
208	7/3/1/3	$P4_232$	4^{+}	group.25-2.1-2	$\mathbb{Z}_2^2$	S_4
209	7/3/2/1	$F432$	$2^{\circ-}$	group.25-1.1-0	$\mathbb{Z}_2$	S_4
210	7/3/2/2	$F4_132$	2^{+}	group.25-1.1-1	$\mathbb{Z}_2$	S_4
211	7/3/3/1	$I432$	$8^{+\circ}$	group.25-3.1-0	$\mathbb{Z}_2^2$	S_4
212	7/3/1/2	$P4_332$	$2^{+}/4$	group.25-2.1-1	$\mathbb{Z}_2$	S_4
214	7/3/3/2	$I4_132$	$4^{+}/4$	group.25-3.1-1	$\mathbb{Z}_2^2$	S_4
215	7/4/1/1	$P\bar{4}3m$	$2^{\circ}{:}2$	group.24-2.1-0	$\mathbb{Z}_2^2$	S_4
216	7/4/2/1	$F\bar{4}3m$	$1^{\circ}{:}2$	group.24-1.1-0	$\mathbb{Z}_2$	S_4
217	7/4/3/1	$I\bar{4}3m$	$4^{\circ}{:}2$	group.24-3.1-0	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	S_4
218	7/4/1/2	$P\bar{4}3n$	4°	group.24-2.1-1	$\mathbb{Z}_4$	S_4
219	7/4/2/2	$F\bar{4}3c$	$2^{\circ\circ}$	group.24-1.1-1	$\mathbb{Z}_2$	S_4
220	7/4/3/2	$I\bar{4}3d$	$4^{\circ}/4$	group.24-3.1-1	$\mathbb{Z}_4$	S_4
221	7/5/1/1	$Pm\bar{3}m$	$4^{-}{:}2$	max.5-2.1-0	$\mathbb{Z}_2^3$	$S_4 \times C_2$
222	7/5/1/3	$Pn\bar{3}n$	$8^{\circ\circ}$	max.5-2.1-1	$\mathbb{Z}_2^2$	$S_4 \times C_2$
223	7/5/1/2	$Pm\bar{3}n$	8°	max.5-2.1-3	$\mathbb{Z}_2^2$	$S_4 \times C_2$
224	7/5/1/4	$Pn\bar{3}m$	$4^{+}{:}2$	max.5-2.1-2	$\mathbb{Z}_2^3$	$S_4 \times C_2$
225	7/5/2/1	$Fm\bar{3}m$	$2^{-}{:}2$	max.5-1.1-0	$\mathbb{Z}_2^2$	$S_4 \times C_2$
226	7/5/2/2	$Fm\bar{3}c$	4^{--}	max.5-1.1-2	$\mathbb{Z}_2^2$	$S_4 \times C_2$
227	7/5/2/4	$Fd\bar{3}m$	$2^{+}{:}2$	max.5-1.1-1	$\mathbb{Z}_2^2$	$S_4 \times C_2$
228	7/5/2/3	$Fd\bar{3}c$	4^{++}	max.5-1.1-3	$\mathbb{Z}_2^2$	$S_4 \times C_2$
229	7/5/3/1	$Im\bar{3}m$	$8^{\circ}{:}2$	max.5-3.1-0	$\mathbb{Z}_2^3$	$S_4 \times C_2$
230	7/5/3/2	$Ia\bar{3}d$	$8^{\circ}/4$	max.5-3.1-1	$\mathbb{Z}_2^2$	$S_4 \times C_2$

1987) number. The column headed BBNWZ gives the Brown–Bülow–Neubüser–Wondratschek–Zassenhaus (Brown *et al.*, 1978) symbol. The column headed HM gives the Hermann–Mauguin symbol (Hahn, 1987). The column headed Conway gives the Conway symbol (Conway, 1992; Conway *et al.*, 2001). The column headed *CARAT* gives the *CARAT* (Opgenorth *et al.*, 1998) symbol. The column headed Abelianization gives the abelianization. Here $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the additive cyclic group of order n. We use exponential notation so that $\mathbb{Z}_2^2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The column headed Holonomy gives the holonomy (point) group. Here C_n is the multiplicative cyclic group of order n, D_n is the dihedral group of order $2n$, A_4 is the alternating group of order 12 and S_4 is the symmetric group of order 24. We use exponential notation so that $C_2^2 = C_2 \times C_2$.

The eleven enantiomorphic three-dimensional space-group pairs are (76 – $P4_1$, 78 – $P4_3$), (91 – $P4_122$, 95 – $P4_322$), (92 – $P4_12_12$, 96 – $P4_32_12$), (144 – $P3_1$, 145 – $P3_2$), (151 – $P3_112$, 153 – $P3_212$), (152 – $P3_121$, 154 – $P3_221$), (169 – $P6_1$, 170 – $P6_5$), (171 – $P6_2$, 172 – $P6_4$), (178 – $P6_122$, 179 – $P6_522$), (180 – $P6_222$, 181 – $P6_422$), (212 – $P4_332$, 213 – $P4_132$). As enantiomorphic space groups are isomorphic, we only list the first group of each pair.

As input for our computer programs, we first used the BBNWZ generators for space groups and relations for point groups given in Brown *et al.* (1978). We then reran our programs with generators and relations provided by *CARAT*, and crosschecked the abelianizations. Finally, all the abelianizations in our tables were checked by hand calculations to further insure the accuracy of our calculations.

The first Betti number, β_1, of a space group depends only on the center of the space group by Theorem 1, and so β_1 depends only on the fixed space of the point group; consequently β_1 depends only on the $\mathbb{Q}$-class of the point group. A $\mathbb{Q}$-class is denoted by the first two numbers of the BBNWZ symbol. By inspection, all the first Betti numbers of the three-dimensional space groups were computed correctly by our computer programs.

The abelianizations of the torsion-free two-dimensional space groups (IT Nos. 1, 4) are well known. These two groups are classified by their abelianizations. The abelianizations of the torsion-free three-dimensional space groups (IT Nos. 1, 4, 7, 9, 19, 29, 33, 76, 144, 169) agree with the tables on page 122 of Wolf (1974). These ten groups are classified by their orientability and abelianizations.

The abelianization of a Coxeter group is easily computed from its Coxeter diagram. In particular, the abelianization of a finitely generated Coxeter group is finite of exponent 2. The only one-dimensional space group that is a Coxeter group is D_∞. The two-dimensional space groups that are Coxeter groups are the reducible group 6-*pmm* ($D_\infty \times D_\infty$), and the irreducible groups 11-*p*4*m*, 14-*p*3*m*1 and 17-*p*6*m*. The three-dimensional space groups that are Coxeter groups are the reducible groups with IT Nos. 47 ($D_\infty \times pmm$), 123 ($D_\infty \times p4m$), 187 ($D_\infty \times p3m1$) and 191 ($D_\infty \times p6m$), and the irreducible groups with IT Nos. 216, 221 and 225. The Coxeter diagram of IT No. 216 is a square with all edge labels 3. The diagram of No. 221 is the linear diagram (4, 3, 4), and the diagram of No. 225 is a star with edge labels 4, 3, 3.

If K is a m-dimensional space group and H is an n-dimensional space group, then K × H is an $(m + n)$-dimensional space group. The consistency of our tables can be checked using the relationship $(\mathrm{K} \times \mathrm{H})_{\rm ab} \cong \mathrm{K}_{\rm ab} \oplus \mathrm{H}_{\rm ab}$.

The two-dimensional space groups that are the direct products of one-dimensional space groups are 1-*p*1 ($C_\infty \times C_\infty$), 3-*pm* ($C_\infty \times D_\infty$) and 6-*pmm* ($D_\infty \times D_\infty$).

The three-dimensional space groups that are the direct product of a one-dimensional space group and a two-dimensional space group are the space groups with IT Nos. 1 ($C_\infty \times p1$), 3 ($C_\infty \times p2$), 6 ($C_\infty \times pm$, $D_\infty \times p1$), 7 ($C_\infty \times pg$), 8 ($C_\infty \times cm$), 10 ($D_\infty \times p2$), 25 ($C_\infty \times pmm$, $D_\infty \times pm$), 26 ($D_\infty \times pg$), 28 ($C_\infty \times pmg$), 32 ($C_\infty \times pgg$), 35 ($C_\infty \times cmm$), 38 ($D_\infty \times cm$), 47 ($D_\infty \times pmm$), 51 ($D_\infty \times pmg$), 55 ($D_\infty \times pgg$), 65 ($D_\infty \times cmm$), 75 ($C_\infty \times p4$), 83 ($D_\infty \times p4$), 99 ($C_\infty \times p4m$), 100 ($C_\infty \times p4g$), 123 ($D_\infty \times p4m$), 127 ($D_\infty \times p4g$), 143 ($C_\infty \times p3$), 156 ($C_\infty \times p3m1$), 157 ($C_\infty \times p31m$), 168 ($C_\infty \times p6$), 174 ($D_\infty \times p3$), 175 ($D_\infty \times p6$), 183 ($C_\infty \times p6m$), 187 ($D_\infty \times p3m1$), 189 ($D_\infty \times p31m$) and 191 ($D_\infty \times p6m$).

4. Theory of the abelianization of space groups

Let Γ be an n-dimensional space group with translation group T and point group Π. In this section, we prove that the exponent of the torsion subgroup of $\Gamma_{\rm ab}$ divides the order of Π. We now explain how $\Gamma_{\rm ab}$ is built up from T and $\Pi_{\rm ab}$. To begin with, we have a short exact sequence of groups and homomorphisms

$$1 \to \mathrm{T} \xrightarrow{i} \Gamma \xrightarrow{p} \Pi \to 1.$$

Exact means that the image of a homomorphism is equal to the kernel of the following homomorphism in the sequence. For example, in the above sequence, the image T of the inclusion map i from T to Γ is the kernel of the projection map p from Γ to Π defined by $p(a + A) = A$. *Short* means a sequence of four group homomorphisms with end groups of order 1. In a short exact sequence the second homomorphism is injective and the third homomorphism is surjective. A space group Γ is called *symmorphic* if the above short exact sequence splits, that is, if there exists a homomorphism $s : \Pi \to \Gamma$ such that $ps : \Pi \to \Pi$ is the identity map. The map s is called a *right inverse* of p.

Let $\mathrm{T}_\Pi = \mathrm{T}/[\Gamma, \mathrm{T}]$ where $[\Gamma, \mathrm{T}] = \{[\gamma, \tau] : \gamma \in \Gamma, \tau \in \mathrm{T}\}$. There is an exact sequence that explains how $\Gamma_{\rm ab}$ is built up from T and $\Pi_{\rm ab}$. By Corollary 6.4 of Chapter VII of Brown (1982), we have an exact sequence of homology groups, with $\mathbb{Z}$ coefficients, and homomorphisms induced by the inclusion i of T into Γ and the projection p of Γ onto Π,

$$H_2(\Gamma) \xrightarrow{p_*} H_2(\Pi) \xrightarrow{\delta} \mathrm{T}_\Pi \xrightarrow{i_*} \Gamma_{\rm ab} \xrightarrow{p_*} \Pi_{\rm ab} \to 0.$$

If Γ is symmorphic, then both p_* maps in the above sequence have right inverses, and so we have a split, short, exact sequence

$$0 \to \mathrm{T}_\Pi \xrightarrow{i_*} \Gamma_{\rm ab} \xrightarrow{p_*} \Pi_{\rm ab} \to 0.$$

Therefore, we have $\Gamma_{\rm ab} \cong \mathrm{T}_\Pi \oplus \Pi_{\rm ab}$. Note that 73 of the 219 three-dimensional space groups are symmorphic. The symmorphic space groups are those whose BBNWZ symbol ends in 1 (or whose *CARAT* symbol ends in 0).

If Π is cyclic, then $H_2(\Pi) = 0$, and so we have a short exact sequence

$$0 \to \mathrm{T}_\Pi \xrightarrow{i_*} \Gamma_{\mathrm{ab}} \xrightarrow{p_*} \Pi_{\mathrm{ab}} \to 0.$$

If $H_2(\Pi)$ is nontrivial, then $i_* : \mathrm{T}_\Pi \to \Gamma_{\mathrm{ab}}$ need not be injective. For example, consider the two-dimensional space group 8-*pgg* considered in §2. The relators $A\tau_1A^{-1}\tau_1$, $A\tau_2A^{-1}\tau_2$, $\beta\tau_1\beta^{-1}\tau_1^{-1}$, $\beta\tau_2\beta^{-1}\tau_2$ of the presentation for Γ imply that T_Π has the Abelian presentation $\langle\tau_1, \tau_2; \tau_1^2, \tau_2^2\rangle$. Therefore $\mathrm{T}_\Pi \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Here $\Pi = C_2 \times C_2$, and by Corollary 5.8 of Chapter V of Brown (1982), we have that $H_2(\Pi) \cong \mathbb{Z}_2$. Now the relator $[A, \beta]\tau_1^{-1}\tau_2$ in the presentation for Γ implies that the generator of $H_2(\Pi)$ is mapped to the element $[\Gamma, \mathrm{T}]\tau_1\tau_2^{-1}$ of T_Π by the connecting homomorphism $\delta : H_2(\Pi) \to \mathrm{T}_\Pi$. Therefore $[\Gamma, \mathrm{T}]\tau_1\tau_2^{-1}$ is in the kernel of $i_* : \mathrm{T}_\Pi \to \Gamma_{\mathrm{ab}}$.

In general, we have the exact sequence

$$\mathrm{T}_\Pi \xrightarrow{i_*} \Gamma_{\mathrm{ab}} \xrightarrow{p_*} \Pi_{\mathrm{ab}} \to 0.$$

Denote the exponent of a finite Abelian group G by $\mathrm{Exp}(G)$ and the torsion subgroup of an Abelian group G by $\mathrm{Tor}(G)$. The last exact sequence implies the next proposition.

Proposition 1. If Γ is an n-dimensional space group, then $\mathrm{Exp}(\mathrm{Tor}(\Gamma_{\mathrm{ab}}))$ divides $\mathrm{Exp}(\mathrm{Tor}(\mathrm{T}_\Pi))\mathrm{Exp}(\Pi_{\mathrm{ab}})$. In particular,

$$\mathrm{Exp}(\mathrm{Tor}(\Gamma_{\mathrm{ab}})) \leq \mathrm{Exp}(\mathrm{Tor}(\mathrm{T}_\Pi))\mathrm{Exp}(\Pi_{\mathrm{ab}}).$$

If Γ is the two-dimensional space group 8-*pgg*, then the exponents of Γ_{ab}, T_Π and Π_{ab} are 4, 2 and 2, respectively. Hence, the upper bound for the exponent of $\mathrm{Tor}(\Gamma_{\mathrm{ab}})$ in Proposition 1 is sharp for this example. More generally, if $\Gamma = pgg \times D_\infty^{n-2}$, then the exponents of Γ_{ab}, T_Π and Π_{ab} are 4, 2 and 2, respectively, and so the upper bound for the exponent of $\mathrm{Tor}(\Gamma_{\mathrm{ab}})$ in Proposition 1 is sharp for this example for all $n \geq 2$.

Proposition 2. Let Γ be an n-dimensional space group. Let $i : Z(\Gamma) \to \Gamma$ and $p : \Gamma \to \Gamma/Z(\Gamma)$ be the natural injection and projection. Then i and p induce a short exact sequence

$$1 \to Z(\Gamma) \xrightarrow{i_*} \Gamma_{\mathrm{ab}} \xrightarrow{p_*} (\Gamma/Z(\Gamma))_{\mathrm{ab}} \to 0.$$

Proof. The short exact sequence

$$1 \to Z(\Gamma) \xrightarrow{i} \Gamma \xrightarrow{p} \Gamma/Z(\Gamma) \to 1$$

induces a homology exact sequence

$$Z(\Gamma) \xrightarrow{i_*} \Gamma_{\mathrm{ab}} \xrightarrow{p_*} (\Gamma/Z(\Gamma))_{\mathrm{ab}} \to 0.$$

The group $(\Gamma/Z(\Gamma))_{\mathrm{ab}}$ is finite by Theorem 3 of Farkas (1975), and so the image of i_* is of finite index in Γ_{ab}. By Theorem 1, we have that $Z(\Gamma)$ is a free Abelian group of rank $\beta_1(\Gamma)$, and so i_* must be injection.

Let K be a set of orthogonal transformations of E^n. Define

$$\mathrm{Fix}(\mathrm{K}) = \{x \in E^n : Cx = x \text{ for all } C \in \mathrm{K}\}.$$

Then Fix(K) is a vector subspace of E^n. We denote the order of a group G by $|G|$.

Theorem 2. Let Γ be an n-dimensional space group with translation group T and point group Π. Let K be a subgroup of Π such that

$$\dim(\mathrm{Fix}(\mathrm{K})) = \beta_1(\Gamma).$$

Then $\mathrm{Exp}(\mathrm{Tor}(\mathrm{T}_\Pi))$ divides $|\mathrm{K}|$. In particular, $\mathrm{Exp}(\mathrm{Tor}(\mathrm{T}_\Pi))$ divides $|\Pi|$, and so

$$\mathrm{Exp}(\mathrm{Tor}(\mathrm{T}_\Pi)) \leq |\Pi|.$$

Proof. As $\mathrm{Fix}(\Gamma) \subseteq \mathrm{Fix}(\mathrm{K})$ and

$$\dim(\mathrm{Fix}(\Gamma)) = \beta_1(\Gamma) = \dim(\mathrm{Fix}(\mathrm{K})),$$

we conclude that $\mathrm{Fix}(\Gamma) = \mathrm{Fix}(\mathrm{K})$.

The point group Π acts on T by conjugation. If $A \in \Pi$ and $b + I \in \mathrm{T}$, then $A(b + I)A^{-1} = Ab + I$. Hence it is natural to replace the multiplicative group T with the isomorphic additive group $\mathrm{T}_{\mathrm{ad}} = \{b \in E^n : b + I \in \mathrm{T}\}$. Likewise we replace $Z(\Gamma)$ with the isomorphic group $Z(\Gamma)_{\mathrm{ad}} = \{b \in E^n : b + I \in Z(\Gamma)\}$.

The action of Π on T corresponds to the natural action of Π on T_{ad} by left multiplication. Now T_Π is isomorphic to $(\mathrm{T}_{\mathrm{ad}})_\Pi = \mathrm{T}_{\mathrm{ad}}/\mathrm{H}$ where H is generated by $\{Ab - b : A \in \Pi \text{ and } b \in \mathrm{T}_{\mathrm{ad}}\}$.

Let $b \in \mathrm{T}_{\mathrm{ad}}$. Then we have

$$|\mathrm{K}|b \equiv \textstyle\sum\{Cb : C \in \mathrm{K}\} \bmod \mathrm{H}.$$

If $A \in \mathrm{K}$, then we have

$$(A - I)\textstyle\sum\{Cb : C \in \mathrm{K}\} = 0.$$

Hence, we have that

$$\textstyle\sum\{Cb : C \in \mathrm{K}\} \in \mathrm{Fix}(\mathrm{K}) \cap \mathrm{T}_{\mathrm{ad}} = \mathrm{Fix}(\Gamma) \cap \mathrm{T}_{\mathrm{ad}} = Z(\Gamma)_{\mathrm{ad}}.$$

Therefore $|\mathrm{K}|b \equiv 0 \bmod \mathrm{H}Z(\Gamma)_{\mathrm{ad}}$. Hence $\mathrm{Exp}(\mathrm{T}_{\mathrm{ad}}/\mathrm{H}Z(\Gamma)_{\mathrm{ad}})$ divides $|\mathrm{K}|$. Now $Z(\Gamma)$ injects into Γ_{ab} by Proposition 2. The injection of $Z(\Gamma)$ into Γ_{ab} factors through T_Π, and so we may regard $Z(\Gamma)$ to be a subgroup of T_Π, and likewise we may regard $Z(\Gamma)_{\mathrm{ad}}$ to be a subgroup of $(\mathrm{T}_{\mathrm{ad}})_\Pi$. Then we have that

$$\mathrm{T}_{\mathrm{ad}}/(\mathrm{H}Z(\Gamma)_{\mathrm{ad}}) = (\mathrm{T}_{\mathrm{ad}})_\Pi/Z(\Gamma)_{\mathrm{ad}}.$$

Hence $\mathrm{Exp}(\mathrm{T}_\Pi/Z(\Gamma))$ divides $|\mathrm{K}|$. As the quotient map from T_Π to $\mathrm{T}_\Pi/Z(\Gamma)$ maps $\mathrm{Tor}(\mathrm{T}_\Pi)$ isomorphically onto a subgroup of $\mathrm{T}_\Pi/Z(\Gamma)$, we have that $\mathrm{Exp}(\mathrm{Tor}(\mathrm{T}_\Pi))$ divides $|\mathrm{K}|$. In particular, $\mathrm{Exp}(\mathrm{Tor}(\mathrm{T}_\Pi))$ divides $|\Pi|$.

If $n = 2$, then Π has an element A such that $\dim(\mathrm{Fix}(A)) = \beta_1(\Gamma)$, and we deduce from Theorem 2 that the exponent of $\mathrm{Tor}(\mathrm{T}_\Pi)$ divides the order of A. For example, let Γ be the symmorphic two-dimensional space group 16-*p*6. Then Π has an element A of order 2 and an element B of order 3 such that $\mathrm{Fix}(A) = \mathrm{Fix}(B) = \{0\}$. Therefore $\mathrm{Exp}(\mathrm{T}_\Pi)$ divides both 2 and 3. Hence $\mathrm{Exp}(\mathrm{T}_\Pi) = 1$, and so $\mathrm{T}_\Pi = \{0\}$.

Let $\tau_i = e_i + I$ for $i = 1, \ldots, n$ be the standard translations, and let $\Gamma = \langle \tau_1, \ldots, \tau_n, -I \rangle$. Then $\mathrm{Exp}(\mathrm{T}_\Pi) = 2 = |\Pi|$, and so the upper bound for the exponent of $\mathrm{Tor}(\mathrm{T}_\Pi)$ in Theorem 2 is sharp for this example for all $n \geq 1$.

Proposition 1 and Theorem 2 imply that $\mathrm{Exp}(\mathrm{Tor}(\Gamma_{\mathrm{ab}}))$ divides $|\Pi|\mathrm{Exp}(\Pi_{\mathrm{ab}})$. The next theorem gives a better result.

Theorem 3. If Γ is an n-dimensional space group with point group Π, then $\mathrm{Exp}(\mathrm{Tor}(\Gamma_{\mathrm{ab}}))$ divides $|\Pi|$, and so

$$\mathrm{Exp}(\mathrm{Tor}(\Gamma_{\mathrm{ab}})) \leq |\Pi|.$$

Proof. For each $A \in \Pi$, choose a coset representative $a_A + A$ of T in Γ corresponding to A. Given an element $b + B \in \Gamma$ and a coset representative $a_A + A$, then

$$(a_A + A)(b + B)(a_{AB} + AB)^{-1} \in \mathrm{T}.$$

The transfer homomorphism $tr : \Gamma \to \mathrm{T}$ is defined by the formula

$$tr(b + B) = \prod\{(a_A + A)(b + B)(a_{AB} + AB)^{-1} : A \in \Pi\}.$$

As T is Abelian, tr induces a homomorphism $tr_* : \Gamma_{\mathrm{ab}} \to \mathrm{T}$. The inclusion map $i : \mathrm{T} \to \Gamma$ induces a homomorphism $i_* : \mathrm{T} \to \Gamma_{\mathrm{ab}}$. The composition $i_* tr_* : \Gamma_{\mathrm{ab}} \to \Gamma_{\mathrm{ab}}$ is obviously multiplication by $|\Pi|$. Now $tr_*(\mathrm{Tor}(\Gamma_{\mathrm{ab}})) = \{0\}$, since T is torsion free. Therefore, we have that

$$|\Pi|\mathrm{Tor}(\Gamma_{\mathrm{ab}}) = i_* tr_*(\mathrm{Tor}(\Gamma_{\mathrm{ab}})) = \{0\}.$$

Hence $\mathrm{Exp}(\Gamma_{\mathrm{ab}})$ divides $|\Pi|$.

Let Γ be a symmorphic n-dimensional space group with cyclic point group Π. Then $\Gamma_{\mathrm{ab}} \cong \mathrm{T}_\Pi \oplus \Pi$. Hence $\mathrm{Exp}(\mathrm{Tor}(\Gamma_{\mathrm{ab}})) = |\Pi|$, and so the upper bound for the exponent of $\mathrm{Tor}(\Gamma_{\mathrm{ab}})$ in Theorem 3 is sharp for this example for all $n \geq 1$.

Let Γ be the three-dimensional space group with IT No. 197. Then $\Gamma_{\mathrm{ab}} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_4$, and the point group Π of Γ is isomorphic to A_4. Hence $\mathrm{Exp}(\Gamma_{\mathrm{ab}}) = 12 = |\Pi|$, and so the upper bound for the exponent of Γ_{ab} in Theorem 3 is sharp for this example.

Remark: Let Γ be an n-dimensional space group with translation group T and point group Π. By the same transfer argument as in the proof of Theorem 3, the exponent of the torsion subgroup of the kth homology group $H_k(\Gamma, \mathbb{Z})$ of Γ divides $|\Pi|$ for each positive integer k, since $H_k(\mathrm{T}, \mathbb{Z})$ is torsion free for each k.

A group Γ is said to be *perfect* if $\Gamma_{\mathrm{ab}} = \{0\}$. In looking over our tables, one sees that every n-dimensional space group is imperfect for $n = 1, 2, 3$. The reason this is true is that the point group Π is solvable for $n = 1, 2, 3$, and so either Π is trivial or Π_{ab} is nontrivial. This is no longer true for $n = 4$. Let Γ be one of the two four-dimensional space groups whose $\mathbb{Z}$-class has BBNWZ symbol 31/3/2. Then Π is isomorphic to A_5, which is a simple group of order 60, and so $\Pi_{\mathrm{ab}} = \{0\}$. From the generators for Π given in Brown *et al.* (1978), we compute that $\mathrm{T}_\Pi = \{0\}$. Hence $\Gamma_{\mathrm{ab}} = \{0\}$ by Proposition 1. These two groups are the only perfect four-dimensional space groups.

By a computer calculation, using the database provided by *CARAT*, we found that in dimension 5 only six of the 222 018 space groups are perfect. The *CARAT* symbols of these perfect space groups are group.1034-3.1-0, group.1039-1.1-0, group.1039-3.1-1, min.169-1.1-0, min.169-1.1-1 and min.169-2.1-1. The point group of the $\mathbb{Q}$-class min.169 is the simple group A_5 of order 60. The point group of the $\mathbb{Q}$-class group.1039 is the simple group A_6 of order 360. The point group of the $\mathbb{Q}$-class group.1034 is the perfect group of order 960, which is a split extension of $\mathbb{Z}_2^4$ by A_5, denoted $A_5 2^{4'}$ by Holt & Plesken (1989).

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